Quiz 3

Date: October 10, 2025

Question 1. 5 points

Identify the error in the following proof, and explain why it is incorrect.

Proposition: The set $\mathbb{Q}$ of rational numbers equals the set $\mathbb{R}$ of real numbers.
Proof: Let R_n be the collection of all real numbers who have only zeros in ther decimal expansion after the nth digit. Then, each R_n is contained in $\mathbb{Q}$ and so $\bigcup_{n\in\mathbb{N}} R_n \subseteq \mathbb{Q}$. But, as we're letting n tend towards infinity, we see that every real number (even those with infinite-length digit expansions) belong to $\bigcup_{n\in\mathbb{N}} R_n$. Thus,

$$\mathbb{R} \subseteq \bigcup_{n\in\mathbb{N}} R_n \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Thus, $\mathbb{Q} = \mathbb{R}$ as desired.

Solution: The issue is that while we are letting n get increasingly large, there is not a real limit here. In particular, $\bigcup_{n\in\mathbb{N}} R_n$ consists of all real numbers which are in R_n for some n. This mean, by definition, that the union still consists only of real numbers which have all zeros in their decimal expansion after *some finite decimal place* n, it doesn't allow for '$n = \infty$'. Thus, in fact, $\bigcup_{n\in\mathbb{N}} R_n = \mathbb{Q}$.

Rubric:

- **(3 pts)** Identifying that the issue is in the claim that every real number is in $\bigcup_{n\in\mathbb{N}} R_n$.
- **(2 pts)** Articulating exactly why this is an issue.

Question 2. 10 points

Let A, B, and C be subsets of S. Prove that

$$A \times (B \triangle C) = (A \times B) \triangle (A \times C).$$

(Recall here that $X \triangle Y = (X - Y) \cup (Y - X)$.)

Solution: Suppose first that (x, y) belongs to $A \times (B \triangle C)$. Then, by definition $x \in A$ and $y \in B \triangle C$. So, $x \in A$ and ($y \in B - C$ or $y \in C - B$). If $y \in B - C$ then $(x, y) \in A \times B$, as $x \in A$ and $y \in B$, but not in $A \times C$ as $y \notin C$, so (x, y) is in $(A \times B) - (A \times C)$. By symmetry, if $y \in C$ then (x, y) in $(A \times C) - (A \times B)$. Thus, in either case we see (x, y) belongs to $(A \times B) \triangle (A \times C) = \big((A \times B) - (A \times C)\big) \cup \big((A \times C) - (A \times B)\big)$.

Conversely, suppose that (x, y) belongs to $(A \times B) \triangle (A \times C)$. If $(x, y) \in A \times B$ but $(x, y) \notin A \times C$, then $x \in A$ and $y \in B$, but also as $x \in A$ the only possibility that $(x, y) \notin A \times C$ is

$y \notin C$. Thus, we see that $x \in A$ and $y \in B - C$. Thus, $(x, y) \in A \times (B \triangle C)$. By symmetry, if $(x, y) \in A \times C$ and $(x, y) \notin A \times B$ then $(x, y) \in A \times (B \triangle C)$.

Thus, we see that $A \times (B \triangle C) \subseteq (A \times B) \triangle (A \times C)$ and $(A \times B) \triangle (A \times C) \subseteq A \times (B \triangle C)$. Thus, $A \times (B \triangle C) = (A \times B) \triangle (A \times C)$, as desired.

Rubric:

- **(2 pts)** Proof-writing coherence.
- **(2 pts)** Having correct strategy (i.e., showing each side is contained in the other).
- **(3 pts)** Correctly showing that the left-hand side is contained in the right-hand side.
- **(3 pts)** Correctly showing that the right-hand side is contained in the left-hand side.